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Asymptotic Expansions of Solutions of $(\nabla^2+k^2)u=0$

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Abstract

A method for constructing asymptotic expansions of solutions of the reduced wave equation $(\nabla^2 + k^2)u = 0$ is derived. The method yields expansions containing exponential decay factors and fractional powers of k, and the possible powers of k in such expansions are determined. The expansions contain a number of undetermined quantities which can be adjusted to yield expansions of solutions of particular problems. The expansions yielded by this method are more general than those given by the Luneburg-Kline method, and possess the behavior exhibited by solutions of various diffraction problems. In the Appendix a theorem on the asymptotic expansions of solutions of general linear equations is derived.

1. Introduction

In many of the diffraction problems which arise in electromagnetic theory, acoustics and other fields it is necessary to determine the behavior as $k \to \infty$ ($k = \frac{2\pi}{\lambda}$, where λ is the wavelength) of solutions of the reduced wave equation

$$(1) \qquad (\nabla^2 + k^2)u = 0.$$

This behavior is usually determined by obtaining exact solutions and expanding them asymptotically in k. If it were possible to obtain the asymptotic expansion of a solution directly from (1) and the other conditions of the problem it might be possible to obtain such expansions in problems for which the exact solutions are not known. It might also be easier to proceed in this way even in problems for which the exact solutions are known.

A direct method for constructing the first term in the expansion of a periodic electromagnetic field was devised by R. K. Luneburg [1], and this method was extended by M. Kline [2] to yield further terms in the expansion and also extended to other equations [11]. This expansion is related to the rays of geometrical optics; the first term in the expansion is called the " geometric optics field" . Derivations of some of the same results have been given by F. G. Friedlander [3], J. Riblet (unpublished), H. Bremmer [4] and E. T. Copson [5]. This method is explained and applied to a number of diffraction problems by J. B. Keller and I. Kay [6]; another application was given by C. Schensted [10].

The above method does not yield all asymptotic expansions of electromagnetic fields. Other kinds of expansions which contain exponential decay factors and fractional powers of k, neither of which appears in the Luneburg-Kline expansions, have occurred in various diffraction problems. Such expansions have been found from the exact solutions for the electromagnetic field due to a dipole near a sphere (H. Bremmer [7]) and for the acoustic field produced near a rigid cylinder by a plane wave (F. G. Friedlander [9]). Similar terms occur in an approximate solution obtained by W. Franz and K. Depperman [8] for the electromagnetic field near a perfectly conducting cylinder due to an incident plane wave.

In this paper we attempt to obtain, by a direct method, more general asymptotic expansions than those given by the Luneburg-Kline method, but we only consider solutions of equation (1). We seek expansions containing exponential decay factors and fractional powers of k, in order to include those expansions found in the problems referred to above. Specifically we take the solution of (1) to be in the form

$$u = v \cdot \exp(ik\phi - k^{\alpha}\chi) \tag{2}$$

and assume that v has the asymptotic expansion

$$v \sim \sum_{n=0}^{\infty} \frac{v_n(x,y,z)}{k^{\lambda_n}} . \tag{3}$$

The functions ϕ, χ and v_n and the real numbers α and λ_n are to be determined in such a way that $\lambda_{n+1} > \lambda_n$ and that equations (2) and (3) satisfy (1) formally. Conditions on all these quantities are obtained, by means of which expansions of the form given in (2), (3) can be constructed. However we do not prove that the expansion so constructed is the expansion of some solution of (1).

In the Appendix to this paper we state and prove a useful theorem about the asymptotic expansions of solutions of linear equations. By means of this theorem the constants λ_n in expansions of the type given in (3) can be determined, and equations for the v_n can be deduced. This theorem is used in our analysis of (3).

2. Derivation of the Asymptotic Expansion

As a first step toward the determination of the expansion given by (2) and (3), we will show that the constant α in (2) may, without loss of generality, be assumed to be positive. To this end we observe that if α is negative or zero, then the Taylor expansion of $e^{-k^{\alpha}\chi}$ in powers of $k^{\alpha}\chi$ proceeds in non-increasing powers of k. Therefore if this factor is expanded, and the resulting series multiplied by the expansion in (3), the product so obtained will also be of the form given by (3). Thus u can be rewritten with a modified v and without the factor $e^{-k^{\alpha}\chi}$. But this expansion, as well as that obtained when $\alpha = 1$, will coincide with the Luneburg-Kline ex-

pansion, as can be seen by applying the theorem in the Appendix. From that theorem it follows that when $e^{-k^{\alpha}\chi}$ is absent or when $\alpha = 1$ all the λ_n are integers, as in the Luneburg-Kline case. Therefore only positive values of α different from unity need be considered in seeking new expansions.

Inserting (2) into (1) yields

$$(4) \quad k^2[1-(\nabla\phi)^2] - 2ik^{1+\alpha}v\nabla\phi \cdot \nabla\chi + ik[2\nabla v \cdot \nabla\phi + v\nabla^2\phi] + k^{2\alpha}v(\nabla\chi)^2 - k^{\alpha}[2\nabla v \cdot \nabla\chi + v\nabla^2\chi] + \nabla^2 v = 0.$$

If $\alpha > 1$, the highest power of k in (4) is 2α and therefore its coefficient may be equated to zero; this yields

$$(5) \qquad (\nabla\chi)^2 = 0.$$

We now assume that χ is real, or that there exists a complex constant b such that $b\chi$ is real, and then (5) implies that χ is a constant. Hence the factor $e^{-k^{\alpha}\chi}$ can be removed from (2) and then (2), (3) again reduce to the Luneburg-Kline expansion. Therefore the case $\alpha > 1$ need not be considered in seeking new expansions, and α is restricted to the range $0 < \alpha < 1$.

With α so restricted, the first term in (4) contains the highest power of k, and upon equating this term to zero we obtain

$$(6) \qquad (\nabla\phi)^2 = 1 .$$

Equation (6) is the eiconal equation of geometrical optics which has been thoroughly analyzed, and which can be used to determine the phase ϕ. After the first term in (4) has been eliminated the second term contains the highest power of k; equating this term, in turn, to zero we have

$$(7) \qquad \nabla\phi \cdot \nabla\chi = 0.$$

Equation (7) states that the equal-amplitude surfaces χ = const. are orthogonal to the equal-phase surfaces ϕ = const. Therefore the surfaces χ = const. contain the rays, which are straight lines orthogonal to the surfaces ϕ = const., and hence these surfaces are ruled or developable.

We will now show that we need not consider the possibility $\alpha > \frac{1}{2}$, for in that case the term $k^{2\alpha}v(\nabla\chi)^2$ would have the highest power of k of the remaining terms. Equating this term to zero would yield equation (5), and the considerations immediately following that equation also enable us to exclude this case. Thus α is restricted to the range $0 < \alpha \leq \frac{1}{2}$.

It is now necessary to distinguish between the two cases $\alpha = \frac{1}{2}$ and $\alpha < \frac{1}{2}$. In the first case the third and fourth terms in (4) contain the same power of k, while in the second case all terms in (4) have distinct powers of k. Using (6), (7) in each of these cases, we obtain instead of (4)

$$(8) \qquad ik\left[2\nabla v \cdot \nabla\phi + v\nabla^2\phi - iv\nabla^2\chi\right] - k^{1/2}\left[2\nabla v \cdot \nabla\chi + v\nabla^2\chi\right] + \nabla^2 v = 0, \qquad \alpha = \frac{1}{2}$$

and

$$(9) \qquad ik\left[2\nabla v \cdot \nabla\phi + v\nabla^2\phi\right] + k^{2\alpha}v(\nabla\chi)^2 - k^{\alpha}\left[2\nabla v \cdot \nabla\chi + v\nabla^2\chi\right] + \nabla^2 v = 0, \; 0 < \alpha < \frac{1}{2}.$$

We now insert the expansion (3) for v into (8) and (9) and assume that it can be differentiated termwise to yield the asymptotic expansions of the derivatives of v. In the resulting expansions the highest powers of k occur in the terms resulting from insertion of the first term in (3) into the first terms in (8) and of (9). Equating each of these terms to zero yields

$$(10) \qquad 2\nabla v_o \cdot \nabla\phi + v_o\nabla^2\phi - iv_o\nabla^2\chi = 0 \qquad \alpha = \frac{1}{2}$$

$$(11) \qquad 2\nabla v_o \cdot \nabla\phi + v_o\nabla^2\phi = 0 \qquad 0 < \alpha < \frac{1}{2}.$$

Both equations (10) and (11) are first-order homogeneous linear differential equations for v_o, and they are in fact ordinary differential equations along the rays, i.e., the orthogonal trajectories of the surfaces ϕ = const., which are straight lines. Equation (11) is the same equation as that satisfied by the first term in the Luneburg-Kline expansion, and it has a simple solution and geometrical interpretation.

In order to determine the constants λ_n and the equations satisfied by the v_n, for $n > 0$, we first assume that no v_n for $n > 0$ satisfies the same equation as does

v_o. If this assumption were not made, then the expansion for v could consist of a combination of any number of expansions of different solutions, each starting with an arbitrary power of k. With this assumption, we may apply the theorem given in the Appendix; this yields the following results:

a) λ_n is the (n+1)-th of the numbers obtained by arranging the numbers $m_1(1-2\alpha) + m_2(1-\alpha) + m_3$ in ascending order, where m_1, m_2, m_3 are any non-negative integers. In the case $\alpha = \frac{1}{2}$, $\lambda_n = \frac{n}{2}$.

b) If $\alpha = \frac{1}{2}$ the coefficients v_n satisfy

$$(12) \qquad 2\nabla v_1 \cdot \nabla\phi + v_1(\nabla^2\phi - i\nabla^2\chi) = -[2\nabla v_o \cdot \nabla\chi + v_o\nabla^2\chi],$$

$$(13) \qquad 2\nabla v_n \cdot \nabla\phi + v_n(\nabla^2\phi - i\nabla^2\chi) = -[2\nabla v_{n-1} \cdot \nabla\chi + v_{n-1}\nabla^2\chi] + i\nabla^2 v_{n-2},\ n \geq 2.$$

c) If $0 < \alpha < \frac{1}{2}$, the coefficients v_n satisfy the equations

$$(14) \qquad 2\nabla v_n \cdot \nabla\phi + v_n\nabla^2\phi = i(\nabla\chi)^2 v_{p_1(n)} - i(2\nabla v_{p_2(n)} \cdot \nabla\chi + v_{p_2(n)}\nabla^2\chi) + i\nabla^2 v_{p_3(n)},$$
$$n > 0.$$

The integers $p_1(n)$, $p_2(n)$ and $p_3(n)$ are defined by the equations

$$(15) \qquad \lambda_n - (1-2\alpha) = \lambda_{p_1(n)}, \quad \lambda_n - (1-\alpha) = \lambda_{p_2(n)}, \quad \lambda_n - 1 = \lambda_{p_3(n)}.$$

If any of these equations has no solution, the corresponding $p_i(n)$ is not defined and the corresponding term is omitted from (14).

To exemplify these results for the case $0 < \alpha < \frac{1}{2}$, we will consider the case $\alpha = \frac{1}{3}$ which occurs in the diffraction problems referred to in the Introduction. In this case we have $1-2\alpha = \frac{1}{3}$, $1-\alpha = \frac{2}{3}$ and therefore $\lambda_n = \frac{n}{3}$. Then from (15) we see that $p_1(1) = 0$ while $p_2(1)$ and $p_3(1)$ are undefined; $p_1(2) = 1$, $p_2(2) = 0$ while $p_3(2)$ is undefined, and for $n \geq 3$, $p_i(n) = n-i$ $(i = 1,2,3)$. Thus from (14) we obtain

(16) $$2\nabla v_1 \cdot \nabla\phi + v_1\nabla^2\phi = i(\nabla\chi)^2 v_o,$$

(17) $$2\nabla v_2 \cdot \nabla\phi + v_2\nabla^2\phi = i(\nabla\chi)^2 v_1 - i(2\nabla v_o \cdot \nabla\chi + v_o\nabla^2\chi),$$

(18) $$2\nabla v_n \cdot \nabla\phi + v_n\nabla^2\phi = i(\nabla\chi)^2 v_{n-1} - i(2\nabla v_{n-2}\cdot\nabla\chi + v_{n-2}\nabla^2\chi) + i\nabla^2 v_{n-3}, \quad n \geq 3.$$

As another example, let $\alpha = \frac{1}{4}$. Then $1-2\alpha = \frac{1}{2}$, $1-\alpha = \frac{3}{4}$ and we find that $\lambda_o = 0$, $\lambda_n = \frac{n+1}{4}$ for $n > 0$. Thus the λ_n form a sequence of multiples of $\frac{1}{4}$, with the term $\frac{1}{4}$ itself omitted. From (15) we find that $p_1(1) = 0$ while $p_2(1)$ and $p_3(1)$ are undefined; $p_2(2) = 0$ while $p_1(2)$ and $p_3(2)$ are undefined; $p_1(3) = 1$, $p_3(3) = 0$ while $p_2(3)$ is undefined; and for $n > 3$, $p_i(n) = n-i-1$ $(i=1,2,3)$. The equation for the v_n may be immediately obtained from (14) with the aid of these results.

3. Conclusion

A procedure has been found for the construction of asymptotic expansions of the form given by (2), (3) which formally satisfy (1). According to this procedure the phase function ϕ may be any solution of the eiconal equation (5). The level surface of the amplitude function χ are generated by the rays, which are straight lines orthogonal to the surfaces ϕ = const.; and the values of χ on these surfaces may be assigned arbitrarily. The constant α must lie in the interval $0 < \alpha \leq \frac{1}{2}$. If $\alpha = \frac{1}{2}$ then $\lambda_n = \frac{n}{2}$ and the v_n can be determined successively from (10), (12) and (13) which are linear ordinary differential equations along the rays. If $0 < \alpha < \frac{1}{2}$ then λ_n is the (n+1)-th of the numbers obtained by arranging the set of numbers $m_1(1-2\alpha) + m_2(1-\alpha) + m_3$ in an increasing sequence, where m_1, m_2 and m_3 are any non-negative integers. In this case v_o can be determined from (11) and then the other v_n can be obtained successively from (14), all of which are also linear ordinary differential equations along the rays.

The arbitrary elements in this construction are the initial values of ϕ and of the v_n on some surface, the values of χ on each ray, and the value of α.

These quantities may be adjusted in order that the expansion correspond to the solution of a particular boundary value problem.

Appendix

Consider a linear operator L and a solution v of the equation

(A1) $$Lv = 0.$$

Suppose that L possesses an asymptotic expansion, or a convergent expansion, with respect to a parameter ε at $\varepsilon = 0$. Thus

(A2) $$L \sim \sum_{i=0}^{\infty} \varepsilon^{\alpha_i} L_i , \qquad \alpha_{i+1} > \alpha_i \geq 0.$$

Here the L_i are also linear operators and the α_i are non-negative real numbers with $\alpha_o = 0$. Then it is possible that v also possesses an asymptotic expansion of the form

(A3) $$v \sim \sum_{j=0}^{\infty} \varepsilon^{\lambda_j} v_j , \qquad \lambda_{j+1} > \lambda_j ;$$

here the λ_j are undetermined real numbers and the v_j are undetermined elements of the same type as v(e.g., functions, vectors, etc.). If v has such an expansion, what values may the λ_j assume, and what equations must the v_j satisfy?

From the linearity and homogeneity of (A1) it is clear that we may multiply v by an arbitrary power of ε without affecting the equation, so obviously λ_o is arbitrary. To obtain conditions on the other λ_j and on the v_j we insert (A2) and (A3) into (A1), assuming that the asymptotic expansion of Lv is obtained by termwise application of the L_i to the v_j. Thus we have

(A4) $$\sum_{\substack{i=o \\ j=o}}^{\infty} \varepsilon^{\alpha_i + \lambda_j} L_i v_j \sim 0.$$

The smallest exponent of ε in (A4) is $\alpha_o + \lambda_o = \lambda_o$; equating the coefficient of this power of ε to zero, we have

(A5) $$L_o v_o = 0.$$

Now we equate the coefficient of $\varepsilon^{\alpha_o+\lambda_k}$ to zero, obtaining

(A6) $$L_o v_k + \sum_{\substack{i,j \\ \alpha_i+\lambda_j=\lambda_k}} L_i v_j = 0, \qquad k > 0.$$

We now assume that none of the v_k for $k > 0$ satisfies (A5), for then it could be the first term in a distinct asymptotic expansion. But then the sum in (A6) must contain at least one term, so there is at least one pair α_i, λ_j that satisfies $\alpha_i + \lambda_j = \lambda_k$. Since α_i is non-negative by assumption, $\lambda_k > \lambda_j$ and therefore $k > j$. Thus each λ_k is the sum of a preceding λ_j and some α_i. As the first λ is λ_o this immediately implies that every λ is of the form $\lambda_o + \sum_{i=o}^{\infty} m_i\alpha_i$ where the m_i are any non-negative integers. Thus the λ_j have been determined and the equations (A5), (A6) have been found for the v_j. The v_j can be found successively from these equations starting with v_o, since the equation for v_k involves only other v_j with $j < k$. Furthermore all of the equations (A6) are of the same form, and only the inversion of L_o is required for their solution.

These results can be collected into the following

<u>Theorem</u>: Let L be a linear operator with the asymptotic expansion (A2) and v be a solution of (A1) with the asymptotic expansion (A3). If the asymptotic expansion of Lv is given by termwise application of the L_i to the v_j and if $L_o v_k \neq 0$ for $k > 0$, then v_o satisfies (A5) and v_k for $k > 0$ satisfies (A6). The constant λ_o is arbitrary but λ_k is the (k+1)-th number in the increasing sequence formed from the set of numbers $\lambda_o + \sum_{i=1}^{\infty} m_i\alpha_i$ where the m_i are any non-negative integers.

In the application of this theorem to the problem considered in the body of this paper, the L_i are differential operators, of which there are three in (8) and four

in (9) and $\varepsilon = k^{-1}$.

It is interesting, and often convenient, to note another way of deriving the asymptotic expansion (A3) for v. To this end we insert (A2) into (A1) and multiply on the left by L_o^{-1}. We then obtain a different asymptotic "equation" for v, namely

$$\text{(A7)} \qquad v \sim \varepsilon^{\lambda_o} v_o - \sum_{i=1}^{\infty} \varepsilon^{\alpha_i} L_o^{-1} L_i v.$$

Here v_o is a solution of (A5) and λ_o is an arbitrary real number. If we now replace v on the right side of (A7) by $\varepsilon^{\lambda_o} v_o$ we obtain an approximate expression for v. If we then replace v on the right side of (A7) by this now approximate expression, and repeat this process indefinitely we obtain

$$\text{(A8)} \qquad v \sim \sum_{n=o}^{\infty} \left(- \sum_{i=1}^{\infty} \varepsilon^{\alpha_i} L_o^{-1} L_i \right)^n \varepsilon^{\lambda_o} v_o.$$

We will now prove that the asymptotic expansion (A3) of v is also obtained by rearranging the terms in (A8) in increasing powers of ε. We begin by equating the expansion in (A3) to that obtained from (A8) which yields

$$\text{(A9)} \qquad \varepsilon^{\lambda_k} v_k = \sum_{\substack{\{m_i\} \\ \lambda_o + \sum m_i \alpha_i = \lambda_k}} e^{\lambda_o + \sum m_i \alpha_i} \prod_{i=1}^{\infty} (-L_o^{-1} L_i)^{m_i} v_o .$$

In (A9) the m_i are any non-negative integers and the summation is over all sets of m_i satisfying the condition indicated.

From (A9) we see that the λ_k determined from (A8) have the same values as were previously found. It is also clear that the v_k satisfy (A6), since if we multiply (A9) by L_o and rearrange the resulting equation we have

$$\text{(A10)} \qquad L_o v_k = \sum_{\substack{s,j \\ \alpha_s + \lambda_j = \lambda_k}} L_s \left[\sum_{\substack{\{m_i\} \\ \lambda_o + \sum m_i^* \alpha_i = \lambda_j}} \prod_{i=1}^{\infty} (-L_o^{-1} L_i)^{m_i^*} v_o \right].$$

In (A10), m_i^* denotes the same set of non-negative integers as does m_i in (A9) except that one positive m, that one with $i = s$, is reduced by unity. From (A9) we see that the bracketed expression is just v_j, so (A10) coincides with (A6); this completes the proof of the statement.

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